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STRESS INTENSITY FACTORS FOR THE ANNULAR CRACK SURROUNDING AN ELASTIC FIBER UNDER TORSION

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Abstract—The axisymmetric problem of an infinitely long fiber perfectly bonded to an elastic matrix which contains an annular crack surrounding the fiber is investigated for the case of torsion field. The problem is formulated as a singular integral equation of the first kind with a Cauchy type kernel using the integral transform technique. The mode III stress intensity factors at the crack tips are presented when (a) the inner crack tip is away from the interface and (b) the inner crack tip is at the interface. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The problems of an elastic solid with a penny shaped crack or an annular crack in tension or torsion have been considered by a number of investigators (Sneddon and Lowengrub, 1969; Collins, 1962; Choi and Shield, 1982; Erguven, 1985; Selvadurai and Singh, 1985; Keer and Watts, 1976). Recently the axisymmetric problems of an infinitely long elastic fiber perfectly bonded to an elastic matrix have been investigated for the case of tension when an annular crack around the interface exists (Wijeyewickrema *et al.*, 1991; Santhanam, 1993). For elastic problems involving an axisymmetric crack, the integral transform method has been employed and the boundary value problem has been converted into the solution of integral equation (Sneddon and Lowengrub, 1969; Keer and Watts, 1976; Erguven, 1985; Selvadurai and Singh, 1985). Application of such integral transform method to torsional problems dealing with an annular crack around the interface is not accomplished yet.

In this paper the axisymmetric problem of an infinitely long fiber embedded in an infinite matrix with an annular crack surrounding the fiber is considered for the case of axisymmetric torsional loading (see Fig. 1). The problem is formulated by means of integral transform and then reduced to a singular integral equation. The resulting equation is solved

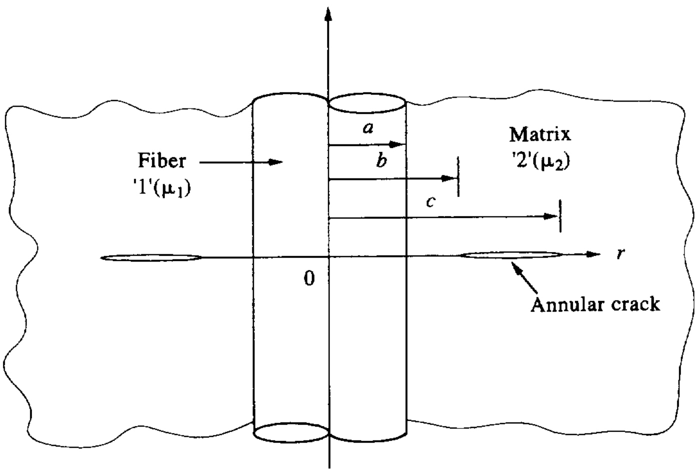


Fig. 1. An annular crack in the matrix surrounding an elastic fiber.

by using the numerical scheme proposed by Erdogan *et al.* (1972) to obtain the relevant physical quantities and the mode III stress intensity factors. The mode III stress intensity factors at the crack tips are presented when (a) the inner crack tip is away from the interface and (b) the inner crack tip is at the interface.

2. DERIVATION OF THE INTEGRAL EQUATION

An infinitely long elastic fiber of radius a is perfectly bonded to an elastic matrix which contains an annular crack surrounding the fiber in $z = 0$ plane, z being the axis of the fiber, as shown in Fig. 1. The inner and outer radii of the annular crack are b and c, respectively, ($b < c < \infty$). A uniform angle of rotation per unit axial length θ_0 is applied to the system at $z = \pm\infty$ and the matrix is unconstrained at $r = \infty$. The required solution is obtained by the superposition of the solution of two related problems. In first problem the perfectly bonded fiber and matrix in the absence of the annular crack are subjected to the uniform angle of rotation per unit axial length θ_0, while the matrix is allowed to deform freely at $r = \infty$. The first problem can be solved without much difficulty and the stress fields are given in Appendix A. The matrix stresses required for the second problem are $\sigma^2_{\theta z}(r, 0) = \mu_2\theta_0 r$ where μ_2 is the shear modulus of the matrix. The stresses applied to the crack surfaces in second problem are those equal and opposite to the stress $\sigma^2_{\theta z}(r, 0)$. Therefore in this paper it is assumed that the crack surface is subjected to quasistatic axisymmetric torsional loads as follows:

$$\sigma^2_{\theta z}(r, 0) = -p(r), \quad b < r < c. \tag{1}$$

The θ-components $u^i_\theta(r, z)$ $(i = 1, 2)$ of the displacement vectors in the fiber and the matrix are the only unknown functions, which satisfy the following differential equations:

$$\frac{\partial^2 u^i_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u^i_\theta}{\partial r} - \frac{u^i_\theta}{r^2} + \frac{\partial^2 u^i_\theta}{\partial z^2} = 0, \quad i = 1, 2 \tag{2}$$

where the superscripts $i = 1$ and $i = 2$ represent the fiber and the matrix zones, respectively. The two nonvanishing components of the stress tensors in the fiber and the matrix are given by

$$\sigma^i_{\theta z}(r, z) = \mu_i \frac{\partial u^i_\theta}{\partial z} \tag{3}$$

$$\sigma^i_{r\theta}(r, z) = \mu_i\left(\frac{\partial u^i_\theta}{\partial r} - \frac{u^i_\theta}{r}\right) \tag{4}$$

where μ_1 and μ_2 are the shear moduli of the fiber and the matrix, respectively. Since $z = 0$ is a plane of symmetry, the semi-infinite domain ($z \geqslant 0$) is considered. Following the method described in Keer and Watts (1976), the solution of eqn (2) for the fiber and the matrix satisfying the condition of regularity at $r = 0$ and $r = \infty$, respectively, may be expressed as

$$u^1_\theta(r, z) = \int_0^\infty A_1(s)e^{-zs}J_1(rs)\,ds + \frac{2}{\pi}\int_0^\infty f_1(s)I_1(rs)\sin zs\,ds \tag{5}$$

$$u^2_\theta(r, z) = \int_0^\infty A_2(s)e^{-zs}J_1(rs)\,ds + \frac{2}{\pi}\int_0^\infty f_2(s)K_1(rs)\sin zs\,ds \tag{6}$$

where $f_1(s), f_2(s)$, $A_1(s)$ and $A_2(s)$ are unknown, $J_n()$ is the Bessel function of the first kind of order n and $I_n()$ and $K_n()$ are the modified Bessel functions of the first and second kinds of order n, respectively. From (3) and (4) the relevant stress components are found to be

$$\sigma^1_{\theta z}(r,z) = \mu_1\left[-\int_0^\infty sA_1(s)e^{-zs}J_1(rs)\,ds + \frac{2}{\pi}\int_0^\infty sf_1(s)I_1(rs)\cos zs\,ds\right] \tag{7}$$

$$\sigma^2_{\theta z}(r,z) = \mu_2\left[-\int_0^\infty sA_2(s)e^{-zs}J_1(rs)\,ds + \frac{2}{\pi}\int_0^\infty sf_2(s)K_1(rs)\cos zs\,ds\right] \tag{8}$$

$$\sigma^1_{r\theta}(r,z) = \mu_1\left[-\int_0^\infty sA_1(s)e^{-zs}J_2(rs)\,ds + \frac{2}{\pi}\int_0^\infty sf_1(s)I_2(rs)\sin zs\,ds\right] \tag{9}$$

$$\sigma^2_{r\theta}(r,z) = \mu_2\left[-\int_0^\infty sA_2(s)e^{-zs}J_2(rs)\,ds - \frac{2}{\pi}\int_0^\infty sf_2(s)K_2(rs)\sin zs\,ds\right]. \tag{10}$$

In the crack problem shown in Fig. 1, the interface conditions between the fiber and the matrix are

$$\sigma^1_{r\theta}(a,z) = \sigma^2_{r\theta}(a,z), \quad u^1_\theta(a,z) = u^2_\theta(a,z) \tag{11}$$

while the plane $z = 0$ is subjected to the conditions

$$\sigma^2_{\theta z}(r,0) = -p(r), \quad b < r < c \tag{12}$$

$$u^1_\theta(r,0) = 0, \quad 0 \leqslant r \leqslant a, \quad u^2_\theta(r,0) = 0, \quad a \leqslant r < b, \quad c \leqslant r < \infty. \tag{13}$$

From eqn (5) and the first eqn of (13), $A_1(s) = 0$. The interface conditions (11) may be used to eliminate two of $f_1(s)$ and $f_2(s)$ and the mixed boundary conditions (12) and (13) may be used to obtain a system of dual integral equations for the unknown $A_2(s)$. Thus, using eqns (5), (6), (9) and (10), the interface conditions (11) may be expressed as

$$\mu_1\frac{2}{\pi}\int_0^\infty sf_1(s)I_2(as)\sin zs\,ds = \mu_2\left[-\int_0^\infty sA_2(s)e^{-zs}J_2(as)\,ds - \frac{2}{\pi}\int_0^\infty sf_2(s)K_2(as)\sin zs\,ds\right] \tag{14}$$

$$\frac{2}{\pi}\int_0^\infty f_1(s)I_1(as)\sin zs\,ds = \int_0^\infty A_2(s)e^{-zs}J_1(as)\,ds + \frac{2}{\pi}\int_0^\infty f_2(s)K_1(as)\sin zs\,ds. \tag{15}$$

Now, rather than substituting $f_2(s)$ given by these two equations into eqn (12) and obtaining a system of dual integral equations for the unknown $A_2(s)$, the problem may be reduced to a singular integral equation in terms of a new unknown function as described in Keer and Watts (1976), defined by

$$G(r) = r\frac{\partial}{\partial r}\left\{\frac{1}{r}u^2_\theta(r,0)\right\}, \quad b < r < c. \tag{16}$$

From eqn (6),

$$G(r) = -\int_0^\infty sA_2(s)J_2(rs)\,\mathrm{d}s. \tag{17}$$

From eqns (6), (13), (16) and (17) it follows that

$$A_2(s) = -\int_b^c tG(t)J_2(st)\,\mathrm{d}t. \tag{18}$$

Taking the Fourier sine transforms of eqns (14) and (15), and using the formulae found in Erdelyi (1954), one is led to the following simultaneous equations for the unknown functions $f_1(s)$ and $f_2(s)$ in terms of the yet undetermined function $G(t)$ as follows:

$$mf_1(s)I_2(as)+f_2(s)K_2(as) = \int_b^c tG(t)I_2(as)K_2(st)\,\mathrm{d}t \tag{19}$$

$$f_1(s)I_1(as)-f_2(s)K_1(as) = \int_b^c tG(t)I_1(as)K_2(st)\,\mathrm{d}t \tag{20}$$

where $m = \mu_1/\mu_2$. Solving eqns (19) and (20), $f_1(s)$ and $f_2(s)$ can be expressed as

$$f_1(s) = \int_b^c tG(t)\frac{\{I_1(as)K_2(as)+K_1(as)I_2(as)\}K_2(st)}{I_1(as)K_2(as)+mI_2(as)K_1(as)}\mathrm{d}t \tag{21}$$

$$f_2(s) = \int_b^c tG(t)\frac{(1-m)I_1(as)I_2(as)K_2(st)}{I_1(as)K_2(as)+mI_2(as)K_1(as)}\mathrm{d}t. \tag{22}$$

From eqns (8), (12) and (22), after substituting for $f_2(s)$ the following integral equation is obtained:

$$\int_b^c\left[\frac{1}{t-r}+k(r,t)\right]G(t)\,\mathrm{d}t = -\frac{\pi p(r)}{\mu_2}, \quad b<r<c \tag{23}$$

where the kernel $k(r,t)$ is given by

$$k(r,t) = k_1(r,t)+2tk_2(r,t) \tag{24}$$

$$k_1(r,t) = \frac{m(r,t)-1}{t-r}+\frac{m(r,t)}{t+r} \tag{25}$$

$$m(r,t) = \begin{cases} \dfrac{2(t^2-r^2)}{rt}[K(r/t)-E(r/t)+\dfrac{r}{t}E(r/t)-\dfrac{t^2-r^2}{tr}K(r/t), & r<t \\ \dfrac{2(t^2-r^2}{t^2}[K(t/r)-E(t/r)]+E(r/t), & r>t \end{cases} \tag{26}$$

$$k_2(r,t) = \int_0^\infty \bar{k}_2(r,t,s)\,\mathrm{d}s \tag{27}$$

$$\bar{k}_2(r,t,s) = sK_1(rs)\frac{(1-m)I_1(as)I_2(as)K_2(st)}{I_1(as)K_2(as)+mI_2(as)K_1(as)} \tag{28}$$

where K and E are the complete elliptic integrals of first and second kind, respectively. The expressions of the infinite integrals used in this part of the analysis are given in Erdelyi (1954). It is seen that for $a \to \infty$, $k_1(r, t)$ vanishes and the integral equation reduces to that of the antiplane shear problem of two bonded elastic half planes containing a crack perpendicular to the interface (Erdogan and Cook, 1974). From eqn (13) and definition (16) it is clear that the integral equation must be solved under the following constraint condition:

$$\int_b^c \frac{G(r)}{r} \mathrm{d}r = 0. \tag{29}$$

The physical significance of eqn (29) is that the crack tips are closed at b and c.

The mode III stress intensity factors at the crack tips are calculated for the two cases; i.e. (a) the inner crack tip is away from the interface and (b) the inner crack tip is at the interface.

2.1. *Inner crack tip away from the interface*

When the inner crack tip is located away from the interface, i.e. $b > a$, a close examination of the kernel $k(r, t)$ defined by eqns (23)–(28) shows that the first part $k_1(r, t)$ has a simple logarithmic singularity of the form $\log|t-r|$, whereas the second part $k_2(r, t)$ is bounded in the closed interval $b \leqslant (r, t) \leqslant c$ provided $a < b < c$. In this case the Cauchy kernel $1/(t-r)$ is the dominant kernel, the index of the integral equation is $+1$ and the solution of eqn (23) has the form (Erdogan *et al.*, 1972)

$$G(r) = g_1(r)[(r-b)(c-r)]^{-1/2} \tag{30}$$

and hence a numerical technique such as that described in Erdogan *et al.* (1972) may be used to determine the unknown function $g_1(r)$ which is bounded in the closed interval $b \leqslant (r, t) \leqslant c$. Thus, defining the following normalized quantities,

$$r = \frac{c-b}{2}\rho + \frac{b+c}{2}, \quad t = \frac{c-b}{2}\tau + \frac{b+c}{2}, \tag{31}$$

$$G(r) = \phi_1(\rho) = F_1(\rho)(1-\rho^2)^{-1/2}, \quad p(r) = \mu_2 P(\rho), \tag{32}$$

$$K(\rho, \tau) = \frac{c-b}{2} k(r, t) \tag{33}$$

eqns (23) and (29) may be expressed as

$$\sum_{i=1}^{n} \frac{1}{n} F_1(\tau_i) \left[\frac{1}{\tau_i - \rho_j} + K(\rho_j, \tau_i) \right] = P(\rho_j), \quad j = 1, \ldots, n-1 \tag{34}$$

$$\sum_{i=1}^{n} \frac{1}{n} \frac{F_1(\tau_i)}{\tau_i(c-b)/2 + (b+c)/2} = 0. \tag{35}$$

where

$$\tau_i = \cos\left(\pi \frac{2i-1}{2n}\right), \quad i = 1, \ldots, n$$

$$\rho_j = \cos\frac{\pi j}{n}, \quad j = 1, \ldots, n-1. \tag{36}$$

After solving eqns (34) and (35) the mode III stress intensity factors which are defined by

$$K_{\rm III}(b) = \lim_{r\to b}\sqrt{2(r-b)}\sigma^2_{\theta z}(r,0),$$
$$K_{\rm III}(c) = \lim_{r\to c}\sqrt{2(r-c)}\sigma^2_{\theta z}(r,0), \tag{37}$$

may be obtained as follows:

$$K_{\rm III}(b) = \lim_{r\to b}\sqrt{2(r-b)}G(r) = \sqrt{(c-b)/2}F_1(-1)$$
$$K_{\rm III}(c) = \lim_{r\to c}\sqrt{2(r-c)}G(r) = -\sqrt{(c-b)/2}F_1(1) \tag{38}$$

where $F_1(-1)$ and $F_1(1)$ are obtained from $F_1(\tau_i)$ $(i = 1,\ldots,n)$ by using the quadratic extrapolation formulas.

2.2. *Inner crack tip at the interface*

For the case $b = c$, i.e. when the inner crack tip is at the interface, $k_2(r,t)$ given by eqn (27) is no longer bounded for all r, t in the closed interval $[b, c]$. It is noted that the kernels of the integral equations are infinite integrals which have a rather slow rate of convergence. The convergence can be improved by subtracting the slowly convergent parts of the integrand which are the leading terms in its asymptotic expansion. These slowly convergent terms can be evaluated in closed form, thereby leading to a rapidly converging infinite integral which can be evaluated numerically along with a closed form expression. By adding and subtracting the asymptotic value of $\bar{k}_2(r,t,s)$ for large values of $s, k_2(r,t)$ may be expressed as the sum of two parts as follows:

$$k_2 f(r,t) = k_2(r,t) + k_{2s}(r,t) \tag{39}$$

where $k_{2f}(r,t)$ is bounded in the corresponding closed interval and becomes unbounded as r and t approach the end point $b = a$. After some manipulations the asymptotic expressions for the integrand and the singular part of the kernel $k_2(r,t)$ are found to be

$$k_{2s}(r,t) = \int_0^\infty \frac{1-m}{1+m}\frac{\mathrm{d}s}{2\sqrt{rt}}\left[\mathrm{e}^{-(r+t-2a)s} - \frac{\alpha(r,t)}{s}(\mathrm{e}^{-(r+t-2a)s} - \mathrm{e}^{-2cs})\right]$$
$$= \frac{1-m}{1+m}\frac{1}{2\sqrt{rt}}\left[\frac{1}{r+t-2a} - \alpha(r,t)\ln\left(\frac{2c}{r+t-2a}\right)\right] \tag{40}$$

where

$$\alpha(r,t) = \frac{3}{8}\left\{\frac{1}{a}\left(4\frac{1-m}{1+m}+6\right) - \frac{5}{t} - \frac{1}{r}\right\}. \tag{41}$$

It should be noted that the singular kernels (40) is essentially the same as the expression found for the corresponding antiplane shear problem considered in Erdogan and Cook (1974) as a, r and $t \to \infty$.

The integral eqn (23) is now of the form

$$\int_a^c \frac{G(t)}{t-r}\,\mathrm{d}t + \int_a^c 2tk_{2s}(r,t)G(t)\,\mathrm{d}t + \int_a^c \{k_1(r,t)+2tk_{2f}(r,t)\}G(t)\,\mathrm{d}t = -\frac{\pi p(r)}{\mu_2}. \tag{42}$$

Equation (42) is a singular integral equation with a generalized Cauchy kernel for which the following solution form is assumed

$$G(r) = g_2(r)(c-r)^{\alpha}(r-a)^{\beta}, \quad a < r < c, \quad -1 < \mathrm{Re}(\alpha,\beta) < 0. \tag{43}$$

Employing the technique given by Erdogan *et al.* (1972), the following characteristic equations are obtained to determine α and β:

$$\cot(\pi\alpha) = 0 \tag{44}$$

$$\cos(\pi\beta) + \frac{1-m}{1+m} = 0. \tag{45}$$

It is noted that eqns (44) and (45) are identical to those obtained in antiplane shear case (Erdogan and Cook, 1974). Equation (44) yields $\alpha = -1/2$ which is well known singularity for crack tip surrounded by a homogeneous medium. The real constant β is a function of the material properties of the fiber and matrix. Normalizing the interval $[a, c]$ by defining

$$r = \frac{c-a}{2}\rho + \frac{a+c}{2}, \quad t = \frac{c-a}{2}\tau + \frac{a+c}{2}, \tag{46}$$

$$G(r) = \phi_2(\rho) = F_2(\rho)(1-\rho)^{\alpha}(1+\rho)^{\beta}, \quad p(r) = \mu_2 P(\rho), \tag{47}$$

$$H_1(\rho,\tau) = \frac{c-a}{2}2tk_{2s}(r,t), \quad H_2(\rho,\tau) = \frac{c-a}{2}\{k_1(r,t)+2tk_{2f}(r,t)\} \tag{48}$$

we obtain

$$\int_{-1}^{1}\left\{\frac{1}{\tau-\rho} + H_1(\rho,\tau) + H_2(\rho,\tau)\right\}F_2(\tau)(1-\tau)^{\alpha}(1+\tau)^{\beta}\,\mathrm{d}\tau = -\pi P(\rho). \tag{49}$$

The constraint condition (29) yields the equation

$$\int_{-1}^{1}\frac{F_2(\tau)(1-\tau)^{\alpha}(1+\tau)^{\beta}}{\tau(c-a)/2+(a+c)/2}\,\mathrm{d}\tau = 0. \tag{50}$$

The singular integral equation with a generalized Cauchy kernel (49) along with the constraint condition (50) is solved using a Gauss–Jacobi integration formula (Erdogan *et al.*, 1972).

The mode III stress intensity factors are defined by

$$K_{\mathrm{III}}(c) = \lim_{r\to c}\sqrt{2(r-c)}\,\sigma_{\theta z}^{2}(r,0), \tag{51}$$

$$K_{\mathrm{III}}(a) = \lim_{r\to a}2^{1/2}(a-r)^{-\beta}\sigma_{\theta z}^{1}(r,0). \tag{52}$$

Making use of the fact that the left-hand-side of eqn (23) yields an expression for $\sigma_{\theta z}^{2}(r,0)$ $(r > c)$, it can be shown that

$$K_{\text{III}}(c) = -2^{1/2}(c-a)^{\beta}g_2(c) = -\lim_{r\to c} 2^{1/2}(c-r)^{-\alpha}G(r) = -2^{1/2+\beta}\{(c-a)/2\}^{1/2}F_2(1) \tag{53}$$

$\sigma^1_{\theta z}(r, 0)$ is obtained from eqns (7) and (21) and the following expression is obtained for $K_{\text{III}}(a)$ as described in Erdogan and Cook (1974).

$$K_{\text{III}}(a) = 2^{1/2}\mu^*(c-a)^{\alpha}g_2(a) = \mu^* \lim_{r\to a} 2^{1/2}(r-a)^{-\beta}G(r) = \mu^*\{(c-a)/2\}^{-\beta}F_2(-1) \tag{54}$$

where

$$\mu^* = \left(\frac{\mu_1}{\mu_2}\right)^{1/2} = (m)^{1/2}. \tag{55}$$

Quadratic extrapolation was used to obtain $F_2(1)$ and $F_2(-1)$.

3. RESULTS AND DISCUSSION

When the inner crack tip is away from the interface, i.e. when both the crack tips have square-root singularities, the normalized mode III stress intensity factors are defined by

$$K'_{\text{III}}(b) = \frac{K_{\text{III}}(b)}{\tau_0 a_1^{1/2}} = F_1(-1), \quad K'_{\text{III}}(c) = \frac{K_{\text{III}}(c)}{\tau_0 \alpha_1^{1/2}} = -F_1(1), \tag{56}$$

where $a_1 = (c-b)/2$. Figures 2 and 3 show the normalized mode III stress intensity factors for the ratios $a/c = 0.5$ and 0.8 under stress distribution $\tau = \tau_0 r/c$ where $\tau_0 = \mu_2\theta_0 c$, respectively. When the size of the crack is very small, i.e. when $b/c \to 1.0$, the mode III stress intensity factors are not sensitive to the presence of the fiber neither are they influenced by the radius of the fiber for all values of m; $K'_{\text{III}}(b)$ and $K'_{\text{III}}(c) \to 1.0$, which is the result for the case of a crack in a homogeneous, isotropic matrix in antiplane shear. When $m > 1$, $K'_{\text{III}}(b) < K'_{\text{III}}(c)$, which implies that the crack would propagate outward from the center. For a given crack size, i.e. for fixed b/c, both $K'_{\text{III}}(b)$ and $K'_{\text{III}}(c)$ decrease with increasing m. To investigate the reduction to the antiplane shear problem (Erdogan and Cook, 1974) for $a \to \infty$, we consider material pairs for which the modulus ratio $m = 23.08$ and 0.043. Figure 4 shows the normalized mode III stress intensity factors for the ratios $a/(c-a) = 10$ and 50, respectively. It is seen that the normalized mode III stress intensity factors approach to that of antiplane shear problem (Erdogan and Cook, 1974) as $a/(c-a)$ increases.

When the inner crack tip is at the interface, the normalized mode III stress intensity factors are defined by

$$K'_{\text{III}}(a) = \frac{K_{\text{III}}(a)}{\tau_0 a_2^{-\beta}} = \mu^* F_1(-1), \quad K'_{\text{III}}(c) = \frac{K_{\text{III}}(c)}{\tau_0 \alpha_2^{1/2}} = -2^{1/2+\beta}F_1(1), \tag{57}$$

where $a_2 = (c-a)/2$. For $m = 1/7$, 1/2, 1.0, 2.0 and 7.0. β takes the values -0.770, -0.608, -0.5, -0.392, -0.230, respectively. The normalized mode III stress intensity factors are given in Fig. 5. $K'_{\text{III}}(a)$ decreases with decreasing m, but the inner crack tip singularity decreases with decreasing m and hence it is not possible to compare $K'_{\text{III}}(a)$ for the different ratios of m. For a given value of m, when the position of the outer crack tip is held fixed, $K'_{\text{III}}(a)$ decreases as the radius of the fiber gets smaller. Since the outer crack tip singularity is independent of m, $K'_{\text{III}}(c)$ increases with decreasing m. When $a/c \to 0$, where c is finite and $a \to 0$, $K'_{\text{III}}(c) \to 4\sqrt{2}/(3\pi)$, which is the solution for the penny shaped crack in a homogeneous, isotropic matrix. For $m = 1.0$ only, $K'_{\text{III}}(c) \to 1.0$ as $a/c \to 1.0$ since $K'_{\text{III}}(c)$ is

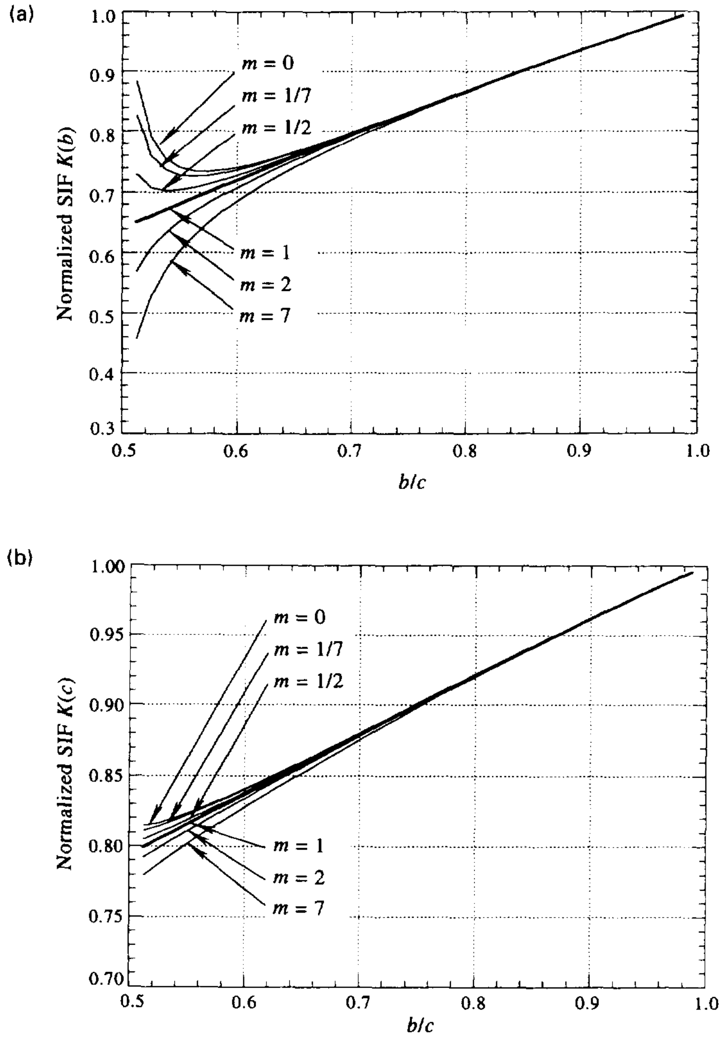


Fig. 2. (a) SIF when the inner crack tip is away from the interface, $a/c = 0.5$; (b) SIF when the inner crack tip is away from the interface, $a/c = 0.5$.

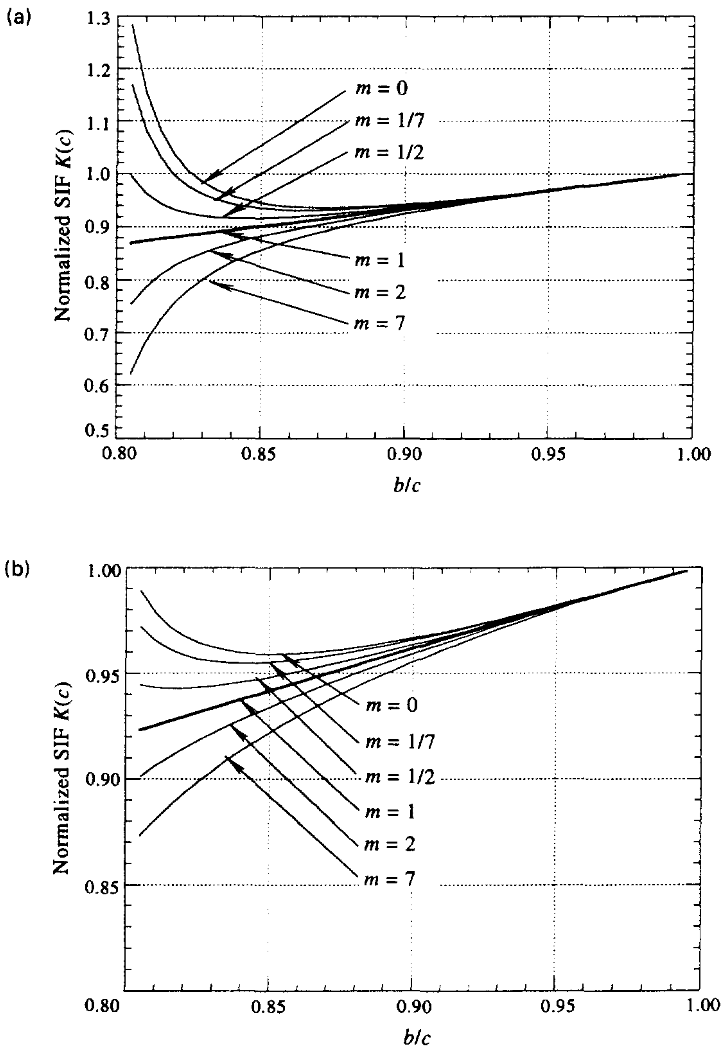
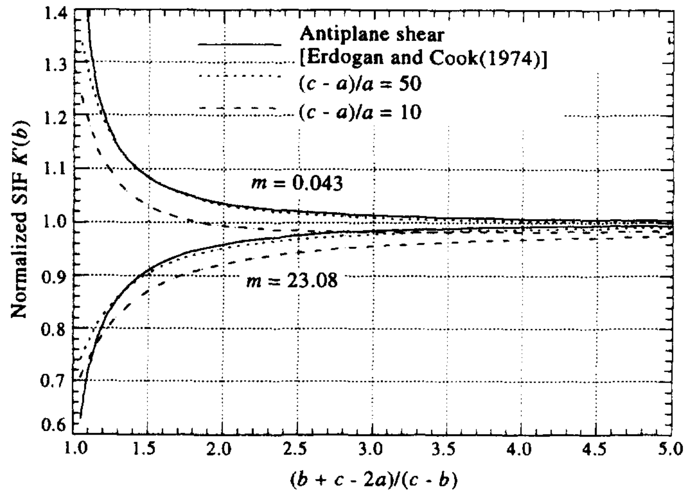


Fig. 3. (a) SIF when the inner crack tip is away from the interface, $a/c = 0.8$; (b) SIF when the inner crack tip is away from the interface, $a/c = 0.8$.

(a)



(b)

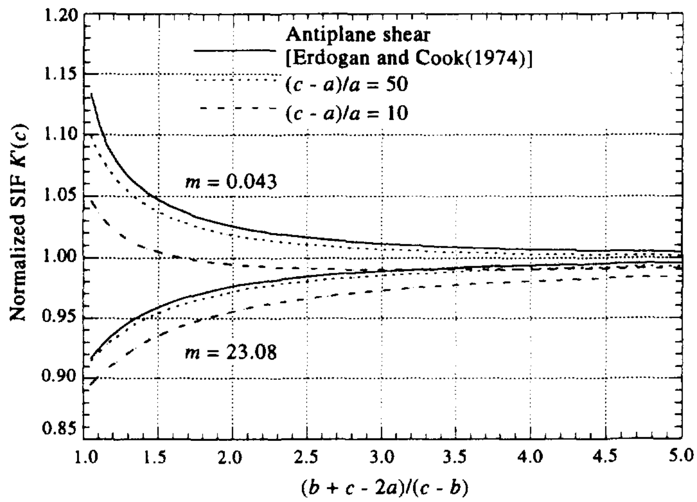


Fig. 4. (a) Comparison of SIF $K'(b)$; (b) comparison of SIF $K'(c)$.

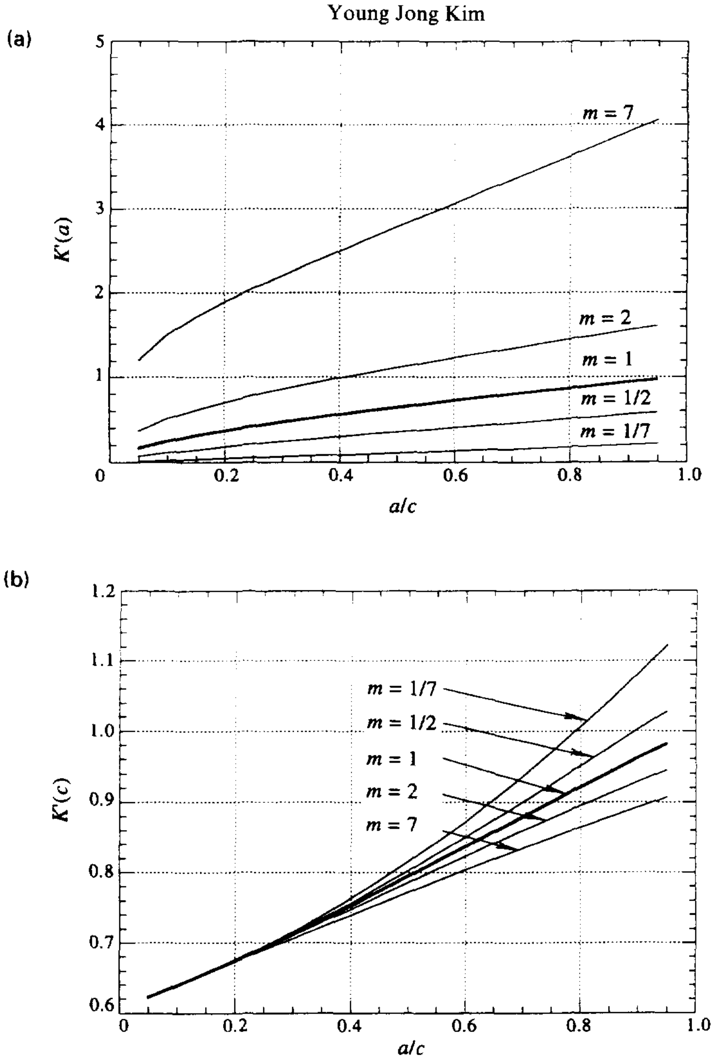


Fig. 5. (a) SIF when the inner crack tip is at the interface; (b) SIF when the inner crack tip is at the interface.

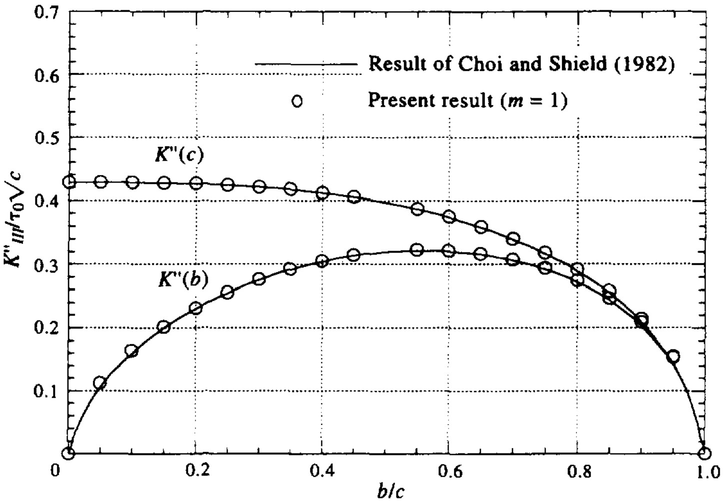


Fig. 6. Comparison with Choi and Shield (1982).

dependent on β as shown in eqn (57). For the comparison with the results of Choi and Shield (1982) for the annular crack in the homogeneous medium, let

$$K''_{\mathrm{III}}(b) = \frac{K_{\mathrm{III}}(b)}{\tau_0 c^{1/2}}, \quad K''_{\mathrm{III}}(c) = \frac{K_{\mathrm{III}}(c)}{\tau_0 c^{1/2}}. \tag{58}$$

Both curves $K''_{\mathrm{III}}(b)$ and $K''_{\mathrm{III}}(c)$ for the case $m = 1.0$ agree well with the results of Choi and Shield (1982) as shown in Fig. 6.

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APPENDIX A

When the fiber and matrix are subjected to a uniform angle of rotation per unit axial length θ_0 at $z = \pm\infty$ and the matrix is unconstrained at $r = \infty$, stress fields can be derived as follows:

$$\sigma^i_{\theta z}(r,z) = \mu_i \frac{\partial u^i_\theta}{\partial z} \quad i = 1,2 \tag{A1}$$

$$\sigma^i_{r\theta}(r,z) = \mu_i\left(\frac{\partial u^i_\theta}{\partial r} - \frac{u^i_\theta}{r}\right) \quad i = 1,2. \tag{A2}$$

Interface conditions at $r = a$

$$\sigma^1_{r\theta}(a,z) = \sigma^2_{r\theta}(a,z) \tag{A3}$$

$$u^1_\theta(a,z) = u^2_\theta(a,z). \tag{A4}$$

Angle of rotation per unit axial length is uniform in each zone regardless of r and z.

$$\theta_i = \frac{\partial}{\partial z}\left(\frac{u^i_\theta(r,z)}{r}\right) \quad i = 1,2 \tag{A5}$$

where θ_i is uniform in each zone.

From these equations it follows that

$$u^i_\theta(r,z) = (\theta_i z + c_i)r, \quad i = 1,2. \tag{A6}$$

Since $z = 0$ is a plane of symmetry,

$$c_i = 0, \quad i = 1,2.$$

From the interface condition (A4),

$$\theta_1 = \theta_2 = \theta_0.$$

From above it follows that

$$u_\theta^i(r, z) = \theta_0 rz, \quad i = 1, 2. \tag{A7}$$

This displacement fields satisfy the interface condition (A3) as follows

$$\sigma_{r\theta}^1(a, z) = \sigma_{r\theta}^2(a, z) = 0.$$

From eqn (A1) it follows that

$$\sigma_{\theta z}^i(r, z) = \mu_i \theta_0 r, \quad i = 1, 2, \tag{A8}$$

where μ_1 and μ_2 are the shear moduli of the fiber and the matrix, respectively.